

UTTG-08-96

hep-th/9606133

June, 1996

F-Theory Duals of Nonperturbative Heterotic $E_8 \times E_8$ Vacua in Six Dimensions

Philip Candelas¹, Eugene Perevalov² and Govindan Rajesh³

Theory Group

Department of Physics

University of Texas

Austin, TX 78712, USA

Abstract

We present a systematic way of generating F-theory models dual to nonperturbative vacua (i.e., vacua with extra tensor multiplets) of heterotic $E_8 \times E_8$ strings compactified on $K3$, using hypersurfaces in toric varieties. In all cases, the Calabi-Yau is an elliptic fibration over a blow up of the Hirzebruch surface \mathbb{F}_n . We find that in most cases the fan of the base of the elliptic fibration is visible in the dual polyhedron of the Calabi-Yau, and that the extra tensor multiplets are represented as points corresponding to the blow-ups of the \mathbb{F}_n .

¹candelas@physics.utexas.edu

²pereval@physics.utexas.edu

³rajesh@physics.utexas.edu

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1 Introduction

In this paper we study six dimensional, $N = 1$, theories arising from compactifications of F-theory on elliptic Calabi-Yau threefolds. Such theories are conjectured to be dual to heterotic $E_8 \times E_8$ strings compactified on $K3$ [24]. Upon further compactification on a torus, this yields [26] the duality between heterotic $E_8 \times E_8$ strings on $K3 \times T^2$ and the type IIA string on the same elliptic Calabi-Yau [1, 2]. This duality has been the subject of extensive study in the recent literature [3 - 14].

The cases studied in [24] were duals to perturbative heterotic vacua on $K3$ with instanton numbers $(12 - n, 12 + n)$ for the E_8 's. Also, it was shown that blowing up the base of the elliptic fibrations produced new models with additional tensor multiplets in six dimensions.

The purpose of this paper is to present a systematic method, using hypersurfaces in toric varieties, of constructing elliptic Calabi-Yau threefolds which provide the F-theory duals of nonperturbative heterotic vacua with arbitrary numbers of tensor multiplets in six dimensions subject to the anomaly cancellation condition [27]

$$d_1 + d_2 + n_T = 25$$

(where d_1 and d_2 are the instanton numbers in the two E_8 's respectively, and n_T is the number of tensor multiplets). Our method generalises the ideas in [24]. We show that there is a simple relation between the distribution of instantons and tensor multiplets on the heterotic side and the Weierstrass equation describing the Calabi-Yau manifold on the F-theory side. Our method differs from the procedure outlined in [18, 20] in that we specify the action of the various \mathbf{C}^* 's on the homogeneous coordinates, instead of adding points to the dual polyhedra of the Calabi-Yau manifolds by hand. In all cases, we find that the Hodge numbers calculated using Batyrev's toric approach [16, 17] are consistent with the results on the heterotic side. Furthermore, we find that the dual polyhedra of these manifolds exhibit the structure described in [18, 20]. We also study Calabi-Yau threefolds describing the F-theory duals of heterotic vacua with an enhanced E_8 gauge symmetry. In such a situation, all the instantons in that E_8 are necessarily small - smoothing out the singularity requires blowing up the base, so that we get as many extra tensors as we had instantons to start with [19].

This paper is organized as follows. In section 2 we give a brief review of F-theory/heterotic duality. In section 3 we describe our procedure for constructing the Calabi-Yau threefolds providing F-theory duals of heterotic vacua with tensor multiplets and arbitrary instanton numbers in the two E_8 's (subject always to the anomaly cancellation condition mentioned above). We study the dual polyhedra of these manifolds and compare them with the polyhedra studied in [18]. We also calculate the Hodge numbers using the methods of Batyrev [16, 17] and find agreement with the results of calculations on the heterotic side. Next, we describe our procedure for constructing duals of heterotic models with enhanced E_8 gauge symmetry. Finally, section 4 summarises our results.

2 F-theory/Heterotic Duality in 6 Dimensions

In this section we give a brief review of the F-theory/heterotic duality in 6 dimensions mainly following [24]. It was shown in [26] that heterotic string theory compactified to 8 dimensions on T^2 is dual to the F-theory compactified on $K3$ which admit an elliptic fibration. The elliptic fiber can be represented by the Weierstrass equation:

$$y^2 = x^3 + ax + b$$

The $K3$ surface can then be viewed as fiber space where the base is \mathbb{P}^1 and the fiber is the torus. The affine coordinate on the \mathbb{P}^1 is denoted by z . The elliptic fibration is specified by

$$y^2 = x^3 + f(z)x + g(z)$$

where f is of degree 8 and g is of degree 12.

If we consider $E_8 \times E_8$ heterotic strings compactified on T^2 with no Wilson lines turned on so that $E_8 \times E_8$ is unbroken, then it's straightforward to show that the F-theory dual is given by the two parameter family (α, β)

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7).$$

Moreover, in the large α and β limit the complex structure of the torus $y^2 = x^3 + \alpha z^4 x + \beta z^6$ is the same as that of the T^2 upon which the heterotic string is compactified. Given this

map we can easily obtain the F-theory encoding of the $K3$ geometry of heterotic string compactification to 6 dimensions. We simply view $K3$ on the heterotic side as an elliptic fibration over \mathbb{P}^1 whose affine coordinate we denote by z' . Now to obtain the corresponding Calabi-Yau threefold on the F-theory side we require α and β to be functions of z' of degree 8 and 12 respectively, thus taking care of the data needed to specify the $K3$ geometry of heterotic strings in the F-theory setup.

Further, to specify the bundle data we make the coefficients of other powers of z functions of z' . Namely, Morrison and Vafa in [24] argued that the F-theory dual for the heterotic string configuration which has d_1 zero size instantons embedded in the first E_8 and d_2 zero size instantons in the second E_8 is given by

$$y^2 = x^3 + f_8(z')z^4x + g_{d_1}(z')z^5 + g_{12}(z')z^6 + g_{d_2}(z')z^7 \quad (1)$$

where the subscripts denote the degree of the polynomial.

If we wish to have no fivebranes [27] on the heterotic side we need to have $d_1 + d_2 = 24$. We will write $d_{1,2} = 12 \pm n$. For this to make sense globally we must take (z, z') to parametrize the rational ruled surface \mathbb{F}_n . Then the deformation away from zero size instantons to a finite size is written in the following form:

$$y^2 = x^3 + \sum_{k=-4}^4 f_{8-nk}(z')z^{4+k}x + \sum_{k=-6}^6 g_{12-nk}(z')z^{6+k} \quad (2)$$

with the condition that negative degree polynomials are set to zero. The moduli with $k < 0$ specify the first E_8 bundle data, and those with $k > 0$ correspond to the second E_8 .

For our purposes, it will be more convenient sometimes to use homogeneous coordinates for the \mathbb{P}^1 's and for the torus, representing the latter as a curve in a weighted projective space $\mathbb{P}_2^{(1,2,3)}$ with homogeneous coordinates w, x, y and a \mathbf{C}^\star acting on them as $(w, x, y) \mapsto (\nu w, \nu^2 x, \nu^3 y)$. The Weierstrass equation in homogeneous coordinates takes the form

$$y^2 = x^3 + f(s, t, u, v)xw^4 + g(s, t, u, v)w^6 \quad (3)$$

where $[s, t]$ and $[u, v]$ are homogeneous coordinates of the base and the fiber \mathbb{P}^1 , respectively, of the \mathbb{F}_n .

Let us also recall some generalities about the spectrum of F-theory compactifications on Calabi-Yau threefolds [24]. Suppose we are given an elliptic Calabi-Yau threefold with Hodge

numbers $h_{11}(X)$ and $h_{21}(X)$ and a base B which has a certain Hodge number $h_{11}(B)$. In six dimensions we will have vector multiplets, hypermultiplets and tensor multiplets whose number we will denote V , H and n_T , respectively.

The scalars in tensor multiplets are in one to one correspondence with the Kähler classes of B except for the overall volume of B which corresponds to a hypermultiplet [26]. Hence, we have

$$n_T = h_{11}(B) - 1 \quad (4)$$

Next, we use the fact that upon further compactification on T^2 the F-theory model in question becomes equivalent to the type IIA on the same Calabi-Yau. Going to the Coulomb phase of vector multiplets in the 4D sense we learn that $H^0 = h_{21}(X) + 1$ and

$$r(V) = h_{11}(X) - h_{11}(B) - 1 \quad (5)$$

where $r(V)$ denotes the rank of the vector multiplets and H^0 denotes the number of neutral hypermultiplets. There is also an anomaly cancellation condition which requires that [28]

$$H - V = 273 - 29n_T \quad (6)$$

where H and V denote the total number of hypermultiplets and vector multiplets respectively.

Note that the base of the elliptic fibration need not be one of the \mathbb{F}_n 's. In fact, that would limit us to having only one tensor multiplet in 6 dimensions (since $h_{11}(\mathbb{F}_n) = 2$). Actually we know [24] that the requirement of having $N = 1$ supersymmetry in six dimensions limit us to a base which is either an Enriques surface, a blowup of \mathbb{P}^2 or \mathbb{F}_n , or a surface with orbifold singularities whose resolution is one of the above. In our construction surfaces which are blowups of \mathbb{F}_n 's will be of importance.

3 Nonperturbative Vacua and Reflexive Polyhedra

3.1 Blowing up the base

Since our task now is to obtain models with more than one tensor multiplet in six dimensions, we need to blow up the base of the elliptic Calabi-Yau in accordance with (4) in Section 2.

The starting point is one of the \mathbb{F}_n models reviewed in Section 2. It is described by the following table:

$$\begin{array}{ccccccc}
 & s & t & u & v & x & y & w \\
 \lambda & 1 & 1 & n & 0 & 2(n+2) & 3(n+2) & 0 \\
 \mu & 0 & 0 & 1 & 1 & 4 & 6 & 0 \\
 \nu & 0 & 0 & 0 & 0 & 2 & 3 & 1
 \end{array} \tag{7}$$

which specifies the exponents with which (λ, μ, ν) act on the homogeneous coordinates s, t, u, v, x, y, w . The Calabi-Yau threefold is obtained by starting with these homogeneous coordinates, removing the loci $\{s = t = 0\}, \{u = v = 0\}, \{x = y = w = 0\}$, taking the quotient by $(\mathbf{C}^*)^3$, and restricting to the solution set of (3) .

3.1.1 $n = 0$

Let's consider the case $n = 0$ which corresponds to symmetric embedding of 24 instantons in the two E_8 's. Our Calabi-Yau threefold in this case has $(h_{11}, h_{21}) = (3, 243)$. We want to blow up one point in the base \mathbb{F}_0 in order to obtain a model with $h_{11}(B) = 2 + 1 = 3$ and $n_T = 2$ according to (4).

To do that we introduce a new coordinate p along with a \mathbf{C}^* which acts on the coordinates as

$$(p, s, t, u, v, x, y, w) \mapsto (\sigma p, \sigma s, t, \sigma u, v, \sigma^6 x, \sigma^9 y, w) \tag{8}$$

We claim that this model is dual to the heterotic one with 12 instantons in the first E_8 and 11 in the second E_8 . Indeed, the equation for the Calabi-Yau threefold in this space looks like the following (Recall that z' and z are affine coordinates of the base and fibre \mathbb{P}^1 's of

the \mathbb{F}_n respectively, while $[s, t]$ and $[u, v]$ are the corresponding homogeneous coordinates.):

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^8 f_{kl} z'^l z^{4+k} p^{8-l-k} x + \sum_{k=-6}^6 \sum_{l=0}^{12} g_{kl} z'^l z^{6+k} p^{12-l-k} \quad (9)$$

We see that certain terms of the original equation are necessarily suppressed by the requirement that we obtain a Calabi-Yau hypersurface here. In particular the highest power of z' multiplying z^5 gives us the number of instantons in the first E_8 , d_1 and that multiplying z^7 supplies us with d_2 . Hence, our model should correspond to $(d_1, d_2) = (12, 11)$, which is in agreement with having $n_T = 2$. We also expect h_{11} of our threefold to go up by 1, and it indeed does, yielding $(h_{11}, h_{21}) = (4, 214)$ which is in perfect agreement with the anomaly cancellation condition (6).

Also note that according to (5)

$$r(V) = 4 - 3 - 1 = 0,$$

and there is no gauge group in six dimensions: only extra tensor multiplet appears in the spectrum. We can continue this process and try to get rid of one more instanton. That can be done by replacing σu in the right-hand side of (8) by $\sigma^2 u$, or by considering the following model:

	p	s	t	u	v	x	y	w
λ	0	1	1	0	0	4	6	0
μ	0	0	0	1	1	4	6	0
ν	0	0	0	0	0	2	3	1
σ	1	1	0	2	0	8	12	0

Indeed, in this case the equation for our threefold takes the form:

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^8 f_{kl} z'^l z^{4+k} p^{8-l-2k} x + \sum_{k=-6}^6 \sum_{l=0}^{12} g_{kl} z'^l z^{6+k} p^{12-l-2k} \quad (10)$$

Again we immediately notice that the terms must be further suppressed by in order to obtain a Calabi-Yau hypersurface. In particular, the maximal power of z' multiplying z^7 is now 10 which tells us that now we have only 10 instantons in the second E_8 on the heterotic side. Ten instantons are enough to break E_8 completely, so we don't expect any gauge group in six dimensions. (5) gives us:

$$r(V) = 5 - 4 - 1 = 0,$$

since the Calabi-Yau now has $(h_{11}, h_{21}) = (5, 195)$.

We can continue this process in an obvious way, namely to obtain the model dual to the heterotic vacuum with $(d_1, d_2) = (12, 12 - q)$ and $n_T = q + 1$ we consider the set of weights:

	p	s	t	u	v	x	y	w
λ	0	1	1	0	0	4	6	0
μ	0	0	0	1	1	4	6	0
ν	0	0	0	0	0	2	3	1
σ	1	1	0	q	0	$2(q+2)$	$3(q+2)$	0

and define the Calabi-Yau exactly as before. The equation now is

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^8 f_{kl} z'^l z^{4+k} p^{8-l-qk} x + \sum_{k=-6}^6 \sum_{l=0}^{12} g_{kl} z'^l z^{6+k} p^{12-l-qk} \quad (11)$$

with the understanding that terms containing negative powers are absent.

We see that requiring that all powers of p are nonnegative amounts to the fact that the maximal power of z' multiplying z^7 in the last sum in the equation is only $12 - q$ (the maximal power of z' multiplying z^5 is still 12) which we interpret as having 12 instantons in the first E_8 and $12 - q$ instantons in the second E_8 in the heterotic dual.

In the same way, we can build duals to vacua with different instanton numbers in the first E_8 also. We simply introduce one more coordinate and one more \mathbf{C}^* which acts on it. Namely, we consider the model defined by the set of weights

	r	p	s	t	u	v	x	y	w
λ	0	0	1	1	0	0	4	6	0
μ	0	0	0	0	1	1	4	6	0
ν	0	0	0	0	0	0	2	3	1
σ	0	1	1	0	q	0	$2(q+2)$	$3(q+2)$	0
ξ	1	0	1	0	0	q'	$2(q'+2)$	$3(q'+2)$	0

(12)

and restrict our set to the solutions of

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^8 f_{kl} z'^l z^{4+k} p^{8-l-qk} r^{8-l+q'k} x + \sum_{k=-6}^6 \sum_{l=0}^{12} g_{kl} z'^l z^{6+k} p^{12-l-qk} r^{12-l+q'k} \quad (13)$$

Again all terms containing negative powers are understood to be nonexistent. In particular having only nonnegative powers of p and r requires that the maximal power of z' multiplying z^5 be $12 - q'$ and that for z^7 be $12 - q$. This, according to our interpretation, shows us that on the heterotic side we have $(d_1, d_2) = (12 - q', 12 - q)$.

3.1.2 $n \neq 0$

Now suppose we had $12 + n$ instantons in the first E_8 and $12 - n$ instantons in the second E_8 in our perturbative heterotic vacuum. The F-theory model dual to it was described by (7) and (2). As before, we want to blow up the base \mathbb{F}_n in order to produce duals to the vacua with more than one tensor multiplets in six dimensions. The way we do it in this case is very similar to how it was done in the previous subsection. Namely, we claim that the model

$$\begin{array}{cccccccc}
& p & s & t & u & v & x & y & w \\
\lambda & 0 & 1 & 1 & n & 0 & 2(n+2) & 3(n+2) & 0 \\
\mu & 0 & 0 & 0 & 1 & 1 & 4 & 6 & 0 \\
\nu & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \\
\sigma & 1 & 1 & 0 & q & 0 & 2(q+2) & 3(q+2) & 0
\end{array} \tag{14}$$

is dual to $(d_1, d_2) = (12 + q, 12 - n)$ heterotic vacuum if $q < n$ and to $(d_1, d_2) = (12 + n, 12 - q)$ if $q > n$. Indeed, the equation in this case has the following form:

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^{8-nk} f_{kl} z'^l z^{4+k} p^{8-l-qk} x + \sum_{k=-6}^6 \sum_{l=0}^{12-nk} g_{kl} z'^l z^{6+k} p^{12-l-qk} \tag{15}$$

So now the maximal power of z' multiplying z^5 is $l_{\max} = \min(12 + n, 12 + q)$ and the maximal power multiplying z^7 is $l_{\max} = \min(12 - n, 12 - q)$, which amounts to our claim.

Similarly, the set of weights

$$\begin{array}{cccccccc}
& p & s & t & u & v & x & y & w \\
\lambda & 0 & 1 & 1 & n & 0 & 2(n+2) & 3(n+2) & 0 \\
\mu & 0 & 0 & 0 & 1 & 1 & 4 & 6 & 0 \\
\nu & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \\
\sigma & 1 & 1 & 0 & 0 & q' & 2(q'+2) & 3(q'+2) & 0
\end{array} \tag{16}$$

with the equation

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^{8-nk} f_{kl} z^l z^{4+k} p^{8-l+q'k} x + \sum_{k=-6}^6 \sum_{l=0}^{12-nk} g_{kl} z^l z^{6+k} p^{12-l+q'k} \quad (17)$$

define the model dual to a nonperturbative heterotic vacuum with $(d_1, d_2) = (12 - q', 12 - n)$.

We are now in a position to put the pieces together and construct a Calabi-Yau threefold which yields F-theory/Type II model dual to a nonperturbative heterotic vacuum with d_1 instantons embedded in the first E_8 and d_2 instantons in the second E_8 for any $d_1 + d_2 < 24$.

If $d_1 > 12$ (we can always assume that $d_2 < 12$) we just take (14) and (15) with either $n = d_1 - 12$ and $q = 12 - d_2$ or vice versa ($q = d_1 - 12$, $n = 12 - d_2$).

If $d_1 < 12$ we take (16) and (17) with $n = 12 - d_2$ and $q' = 12 - d_1$. Note that we could also achieve this result by starting with $n = 0$ and applying (12) and (13) with $q = 12 - d_2$ and $q' = 12 - d_1$.

3.2 Reflexive Polyhedra and Hodge Numbers

Here we will analyze the manifolds constructed in the previous section. Our main tool is Batyrev's toric approach. To a Calabi-Yau manifold defined as a hypersurface in a weighted projective space one can associate its Newton polyhedron, which we denote by Δ . If it happens to be reflexive, which it often (perhaps always) does, we may define the dual or polar polyhedron which we denote by ∇ . By means of a computer program we have computed the dual polyhedra and Hodge numbers for all the examples from the previous section. The polyhedra exhibit a nice regular structure similar to that discussed in [18].

To illustrate the aforementioned structure, let's have a look at the first three cases. Namely, consider the dual polyhedra for the models dual to the heterotic vacua with (d_1, d_2) equal to $(12, 12)$, $(12, 11)$ and $(12, 10)$. The points of dual polyhedra are displayed in Table 3.1.

(12,12)	(12,11)	(12,10)
(-1, 0, 2,-1)	(-1, 0, 2,-1)	(-1, 0, 2,-1)
(0,-1, 1,-1)	(0,-1, 1,-1)	(0,-1, 1,-1)
(0, 0,-1,-1)	(0, 0,-1,-1)	(0, 0,-1,-1)
(0, 0, 0,-1)	(0, 0, 0,-1)	(0, 0, 0,-1)
(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
(0, 0, 1,-1)	(0, 0, 1,-1)	(0, 0, 1,-1)
(0, 0, 1, 0)	(0, 0, 1, 0)	(0, 0, 1, 0)
(0, 0, 1, 1)	(0, 0, 1, 1)	(0, 0, 1, 1)
(0, 0, 1, 2)	(0, 0, 1, 2)	(0, 0, 1, 2)
(0, 1, 1,-1)	(0, 1, 1,-1)	(0, 1, 1,-1)
(1, 0, 0,-1)	(1, 0, 0,-1)	(1, 0, 0,-1)
	(1,-1, 0,-1)	(1, 1, 0,-1)
		(1, 2, 0,-1)

Table 3.1: The dual polyhedra for given (d_1, d_2) on the heterotic side.

The following observations summarize the structure of the polyhedra:

1. For each polyhedron the points from the second up to tenth (counting from above) lie in the hyperplane $x_1 = 0$ and are the same. These points themselves form a reflexive polyhedron which is dual of the polyhedron for the $K3$ of the original fibration.
2. The points from the third up to ninth in each case form a two- dimensional reflexive polyhedron, ${}^2\nabla$, which is a triangle. This ${}^2\nabla$ is the dual polyhedron of the torus $\mathbb{P}_2^{(1,2,3)}$ of the elliptic fibration.
3. The first eleven points in all three cases are exactly the same. The second polyhedron differs from the first by the addition of the point $(1,-1, 0,-1)$ which lies ‘above’ the hyperplane of the $K3$. The third polyhedron differs from the first by the addition of two points ‘above’ the $K3$.

We have checked that this pattern persists in all other examples.¹ Namely, when we blow up one more point in the base of the elliptic fibration thus decreasing the total number of instantons on the heterotic side by one, one more point appears ‘above’ the $K3$ hyperplane. These points always line up, that is they have the form²

$$(1, k, 0, -1), \quad k = 0, 1, \dots, n_T - 1$$

with a few exceptions (e. g. in $(12, 11)$ case we have points $(1, -1, 0, -1)$ and $(1, 0, 0, -1)$). This is in precise agreement (up to a change of basis) with the observation of [18].

Also as d_1 or d_2 become less than 10 the three-dimensional polyhedron which we associate to the $K3$ of the fibration begins to acquire additional points signaling the appearance of unbroken gauge groups which have perturbative interpretation on the heterotic $E_8 \times E_8$ side. This is again in agreement with what we would expect from the heterotic point of view. Indeed, 9 instantons and less is not enough to break E_8 completely, and some subgroup of either (or both) of the E_8 ’s should appear in the perturbative part of the spectrum, which we know is visible in the generic $K3$ fibre of the $K3$ fibration on the Type II side [11]. Moreover, the group which we observe in the $K3$ is precisely the commutant of the maximal subgroup of E_8 which can be completely broken by the given number of instantons, i.e. we are dealing with maximally Higgsed case.

It is interesting to note that projection of the points in Table 3.1 onto the first two coordinates yields the fan of the base of the elliptic fibration - thus in the case of $(12, 12)$ instantons in the two E_8 ’s, we get the points $(1, 0), (0, 1), (0, 0), (-1, 0), (0, -1)$, which we

¹In the cases where both d_1 and d_2 are less than 10, and so there are unbroken subgroups in both E_8 ’s, the subpolyhedron defined by $x_1 = 0$ is still the dual of a $K3$ manifold, but the way the elliptic fibration structure is visible inside it is trickier. Namely, the two-dimensional subpolyhedron which here lies in the plane $x_1 = 0, x_2 = 0$ and which is a dual polyhedron of a torus gets tilted in some cases and is no longer visible in other cases.

²Note that these points are all evenly spaced and lie in a straight line. It is easily verified that they actually lie on an edge of the dual polyhedron. Following [21], it may be argued that such a situation signals the appearance of an enhanced gauge symmetry. However, this argument applies only when the genus of the corresponding curve of singularities is greater than 1. In our case, the genus of this curve is exactly one, as can be seen by going to the original polyhedron and counting the number of integral points interior to the 2-face dual to this edge.

recognise as the fan of \mathbb{F}_0 . Creating a tensor multiplet by removing an instanton, we find that the base acquires an additional point, namely $(1, -1)$. To see that this corresponds to a blowup of \mathbb{F}_0 , we use a standard result in toric varieties[15] - insertion of the sum of two adjacent vectors in the fan of a nonsingular toric surface yields a blowup of the surface. In our case, we simply note that $(1, -1) = (1, 0) + (0, -1)$, so that we do indeed have \mathbb{F}_0 blown up at one point. We find this pattern to hold in all the cases we have studied. Furthermore, we can compute h_{11} of the base, which we know from [24] to be $n_T + 1$. Once again we find exact correspondence between geometry and physics. For \mathbb{F}_n and blowups thereof, $h_{11} = b_2$. The formula for b_2 of a toric surface is $b_2 = d_1 - 2d_0$, where d_i is the number of i dimensional cones. Thus for \mathbb{F}_0 , the point $(0, 0)$ generates the only zero dimensional cone in the fan while all the other points generate one dimensional cones, so we find that $b_2(\mathbb{F}_0) = 4 - 2 = 2$, which is one more than the number of tensor multiplets in the $(12, 12)$ model. Also, blowing up the base increases the number of one dimensional cones by one while leaving the number of zero dimensional cones unchanged, so that b_2 of the base increases by one each time.

Having thus outlined the general picture let's look at some of the examples.

1. $(d_1, d_2) = (12, 12)$. $(h_{11}, h_{21}) = (3, 243)$ As we've already seen, there is only one point in the dual polyhedron above the $K3$ hyperplane, which corresponds to $n_T = 1$ as becomes to a perturbative vacuum. $h_{11}(B) = n_T + 1 = 2$, $r(V) = h_{11}(X) - h_{11}(B) - 1 = 0$, so there is no gauge group in 6D. Indeed, the three-dimensional dual polyhedron corresponding to the generic fiber of the fibration shows no sign of a gauge group according to the results of [18].

If we count the moduli specifying the gauge bundle data, we obtain 114 for each E_8 . At this point, though, we still have the freedom of one rescaling for z [22]. Also, one of the moduli corresponds to a deformation of \mathbb{F}_0 to \mathbb{F}_2 . Thus we have 112 degrees of freedom, which is exactly the quaternionic dimension of the moduli space of 12 E_8 instantons.

2. $(d_1, d_2) = (12, 11)$. $(h_{11}, h_{21}) = (4, 214)$ One point is added 'above' the $K3$, signalling the appearance of one additional tensor multiplet. $h_{11}(B) = 3$, $r(V) = 4 - 3 - 1 = 0$. The $K3$ polyhedron hasn't changed, so we don't expect any perturbative heterotic

contributions to the gauge group. On the other hand, one less instanton results in a tensor multiplet in 6D, which becomes a vector multiplet in 4D. So we expect no nonperturbative heterotic contribution to the 6D gauge group, which is in agreement with what we find for $r(V)$. Also $H = H^0 = 215$, $V = 0$, and the anomaly cancellation condition (6) is satisfied.

The number of moduli corresponding to the first E_8 is, of course, the same as in the previous example. However, the number of moduli specifying the second E_8 bundle has dropped to 83 due to the suppression of certain terms which would yield negative powers of p in (11). Taking into account the z scaling freedom makes it 82, which is the dimension of the moduli space of 11 E_8 instantons.

3. $(d_1, d_2) = (12, 10)$. $(h_{11}, h_{21}) = (5, 185)$. $K3$ is still the same. We see three points ‘above’ which implies $n_T = 3$, $h_{11}(B) = 4$. $r(V) = 5 - 4 - 1 = 0$, $H = H^0 = 186$, $V = 0$, anomaly cancellation obviously holds. We now obtain 54 moduli specifying the bundle data in the second E_8 . However, one of the moduli corresponds to a deformation of \mathbb{F}_2 to \mathbb{F}_0 . There is also one scaling degree of freedom for z , so that the number of degrees of freedom is 52, which is exactly the dimension of the moduli space of 10 E_8 instantons.
4. $(d_1, d_2) = (12, 9)$. $(h_{11}, h_{21}) = (8, 164)$. 4 points correspond to $n_T = 4$, $h_{11}(B) = 5$, and hence $r(V) = 8 - 5 - 1 = 2$. Thus, we expect a rank 2 perturbative (on the heterotic side) gauge group. Indeed, 9 instantons break an E_6 subgroup completely leaving us with an unbroken $SU(3)$, which can also be read off from the $K3$ dual polyhedron. Also, the $SU(3)$ comes with no charged matter in this case, so $H = H^0 = 165$, $V = \dim(SU(3)) = 8$, and the anomaly cancellation reads $165 - 8 = 273 - 29 \cdot 4$.

Let’s also see what the moduli counting tells us. $k > 0$ terms in (11) with $q = 3$ now number 30 (where the rescaling freedom has been taken into account). 9 instantons on the heterotic side fit into an E_6 , which has dual Coxeter number 12 and dimension 78. So, the moduli space of 9 instantons embedded in the E_6 has quaternionic dimension $hd_2 - \dim(G) = 12 \cdot 9 - 78 = 30$ in complete agreement with our number of complex moduli on F-theory side.

5. $(d_1, d_2) = (12, 8)$. $(h_{11}, h_{21}) = (11, 155)$. One more point adds up above the $K3$ hyperplane. Points are also added to the $K3$ itself. Using the results of [18], we see an unbroken $SO(8)$ appearing. Also, we have $n_T = 5$, $h_{11}(B) = 6$, $r(V) = 11 - 6 - 1 = 4$, in agreement with the rank of $SO(8)$. Again, we get no charged matter. Hence, $H = H^0 = 156$, $V = \dim(SO(8)) = 28$, and $156 - 28 = 273 - 5 \cdot 29$ as required by the anomaly cancellation.

The equation (11) now has 20 moduli with $k > 0$. On the other hand, 8 instantons can break an $SO(8)$ subgroup of the second E_8 completely. $SO(8)$ has dual Coxeter number 6 and dimension 28. So, for the dimension of the moduli space of 8 $SO(8)$ instantons we have $6 \cdot 8 - 28 = 20$, again precisely matching the number of F-theory moduli.

6. $(d_1, d_2) = (12, 7)$. $(h_{11}, h_{21}) = (12, 150)$. In the same way, we get $n_T = 6$, $h_{11}(B) = 7$. The shape of the three-dimensional polyhedron corresponding to $K3$ is exactly that identified with the presence of an unbroken E_6 in [18]. We have to be more careful, though. In [29], [22] it was shown that the E_6 singularity can also correspond to an unbroken F_4 . We know also that on the heterotic side breaking can proceed up to F_4 [23]. On the other hand, we obtain $r(V) = 12 - 7 - 1 = 4$, which is in agreement with having an unbroken F_4 in 6D. In addition, the anomaly cancellation condition gives us (supposing $H = H^0$): $151 - 52 = 273 - 29 \cdot 7$ ($\dim(F_4) = 52$), which is in precise agreement with having a matter-free F_4 .

Counting the number of $k > 0$ moduli in (11) with $q = 5$ now yields 14. 7 instantons fit in a G_2 subgroup of the E_8 . G_2 has dual Coxeter number 4 and dimension 14 giving us $4 \cdot 7 - 14 = 14$ for the moduli space of 7 instantons on the heterotic side.

7. $(d_1, d_2) = (12, 6)$. $(h_{11}, h_{21}) = (15, 147)$. One more point above $K3$ appears. All other points are exactly as in $(12, 7)$ case. That is, the subpolyhedron corresponding to the $K3$ still shows us the presence of an E_6 singularity in the Higgs branch. This time, though, we know from [24], [29], [22] and from the argument on the heterotic side that an E_6 gauge group actually makes its appearance. Indeed, we have $n_T = 7$, $h_{11}(B) = 8$, and, hence, $r(V) = 15 - 8 - 1 = 6$ and, moreover, anomalies cancel precisely when

$H = H^0 = 148$ and $V = \dim(E_6) = 78$.

Here we used (11) with $q = 6$ which has 10 $k > 0$ complex moduli. 6 instantons, on the other hand, fit into an $SU(3)$ subgroup which has dual Coxeter number 3 and dimension 8 yielding thus $3 \cdot 6 - 8 = 10$ quaternionic moduli on the heterotic side and showing precise agreement.

8. $(d_1, d_2) = (12, 5)$. $(h_{11}, h_{21}) = (17, 145)$. We expect to see an unbroken E_7 with half a 56 hypermultiplet [24]. Our polyhedron acquires one more point above the $K3$ hyperplane, as well as a certain number of points in the $K3$ subpolyhedron itself, which we can see to signal the appearance of E_7 [18]. We have $n_T = 8$, $h_{11}(B) = 9$ and $r(V) = 17 - 9 - 1 = 7$. Moreover, the anomaly cancellation condition works precisely when $H = H^0 + H^c = 146 + \frac{1}{2} \cdot 56 = 174$ and $V = \dim(E_7) = 133$, which is perfectly consistent with what we expected.

As to the number of moduli, we get 7 of them from our equation (counting only those with $k > 0$) which matches exactly what we obtain on the heterotic side for 5 instantons: they fit in $SU(2)$, which has dual Coxeter number 2 and dimension 3, producing $2 \cdot 5 - 3 = 7$ quaternionic moduli. B

9. $(d_1, d_2) = (12, 4)$. $(h_{11}, h_{21}) = (18, 144)$. In comparison to the previous case, the polyhedron gets one more point above the $K3$. All the rest carry over. So now $n_T = 9$, $h_{11}(B) = 10$. We expect E_7 without any charged matter. Indeed, the $K3$ subpolyhedron is the same as in $(12, 5)$ case and corresponds to E_7 , and for the rank of the gauge group we obtain $r(V) = 18 - 10 - 1 = 7$. Again the 6D anomalies cancel exactly when $H = H^0 = 145$ and $V = \dim(E_7) = 133$.

The equation (11) with $q = 8$ yields 5 moduli. 4 instantons fit in an $SU(2)$ subgroup of the second E_8 . So we obtain $2 \cdot 4 - 3 = 5$ moduli on the heterotic side also.

10. $(d_1, d_2) = (12, 0)$. $(h_{11}, h_{21}) = (23, 143)$. Now there are 13 points above the $K3$ subpolyhedron meaning that $n_T = 13$ and $h_{11}(B) = 14$. The $K3$ subpolyhedron itself acquires some additional points also exhibiting the structure which we claim corresponds to E_8 . For the rank of the gauge group we obtain $r(V) = 23 - 14 - 1 = 8$,

and the anomalies will cancel if $H = H^0 = 144$ and $V = \dim(E_8) = 248$, which is in accord with having matter-free E_8 . This again agrees perfectly with our expectations.

The counting of moduli now is trivial and gives 0 on both sides.

Using the blowup procedure described in the previous subsection, we can also construct vacua dual to heterotic ones with $d_1 < 12$ as well as with $d_1 > 12$. In all of these cases, the rank of the total gauge group obtained from (5) is in precise agreement with what can be deduced from the heterotic side (assuming maximal possible Higgsing). Also, the Hodge numbers work in such a way that the anomaly cancellation condition is always satisfied. Moduli counting shows perfect agreement as well. This can be regarded as a strong check of the proposed algorithm of constructing F-theory duals to nonperturbative heterotic vacua.

3.3 Unhiggsing E_8

While describing various examples in the previous subsection, we left aside the cases when either d_1 or d_2 (or both) become less than 4. We know that in this situation, the instantons cannot be of finite size (various D -terms in six dimensions wouldn't let us inflate them) [25]. This corresponds to the fact that the unbroken gauge symmetry is E_8 . Let's see what happens if we try to consider models with $d_2 < 4$.

1. $(d_1, d_2) = (12, 3)$. In this case our threefold has $(h_{11}, h_{21}) = (23, 143)$ exactly as in $(12, 0)$ case. The manifold itself is apparently different, though: there are 3 fewer points above the three-dimensional subpolyhedron which we associate to $K3$, and those points are not being erased from a codimension one face. The rest of the points stay exactly where they were leaving the three-dimensional subpolyhedron unchanged. We are thus led to the conclusion that unbroken E_8 is present (indeed, zero size instantons can't break it) and also that there appear to be only 10 tensor multiplets in 6 dimensions (again, one tensor multiplet corresponding to one point with $x_1 = 1$). For the rank of the total gauge group we obtain: $r(V) = 8$, and $h_{11}(B) = 23 - 8 - 2 = 13$ strongly suggesting that there are actually 13 tensor multiplets - so that there are 3 extra tensors which are somehow not visible as points in the polyhedron.

It's interesting also to look at the moduli on the F-theory side and see that their number is consistent with having 3 instantons in the second E_8 . Indeed, in (11) with $q = 9$ there are only 4 nonzero terms with $k > 0$ (they are g_{1l} for $l = 0, 1, 2, 3$), 1 of which is irrelevant. This gives us 3 moduli corresponding to the positions of 3 (point-like) instantons on the $K3$ on the heterotic side.

2. $(d_1, d_2) = (12, 2)$. Again we see $(h_{11}, h_{21}) = (23, 143)$ The polyhedron is exactly as before except for one point with $x_1 = 1$ which joins the 10 present in $(12, 3)$ case making their number 11. Thus it appears that $n_T = 11$, $h_{11}(B) = 12$ and $r(V) = 8$, yielding a discrepancy of 2 tensor multiplets.

Exactly as in the previous example, we have 2 moduli on the F-theory side corresponding to the positions of 2 (small) instantons.

3. $(d_1, d_2) = (12, 1)$. Same story: $(h_{11}, h_{21}) = (23, 143)$, $n_T = 12$, $h_{11}(B) = 13$ and $r(V) = 8$, so that we are missing one tensor multiplet.

In the above examples, the instantons happen to be small because they are not allowed to be of finite size. Following [29], we know that we can't have smooth Calabi-Yau's corresponding to small instantons. In the case of small $SO(32)$ instantons, one can smooth out the singularity without blowing up the base, but for small E_8 instantons, smoothing out the singularity necessarily involves blowing up the base - thus the small instantons become tensor multiplets.

We were able to construct an algorithm similar to that of the previous section which allows us to build duals to heterotic vacua with enhanced E_8 gauge symmetry¹. It works as follows.

Given a manifold constructed as described before, add one more new coordinate along with an additional \mathbf{C}^* which acts on the coordinates as:

$$(a, u, x, y) \mapsto (\rho a, \rho^6 u, \rho^{14} x, \rho^{21} y) \tag{18}$$

¹This procedure naturally shrinks all the instantons in the given E_8 .

the action on other coordinates being trivial. Then remove suitable loci and restrict to the set of solutions of

$$y^2 = x^3 + \sum_{k=-4}^4 \sum_{l=0}^{12-nk} f_{kl} z^l z^{4+k} \dots a^{4-6k} x + \sum_{k=-6}^6 \sum_{l=0}^{12-nk} g_{kl} z^l z^{6+k} \dots a^{6-6k} \quad (19)$$

where \dots stand for any other coordinates which can be present.

We claim that this represents the dual to a heterotic vacuum where the second E_8 is unhiggsed. Indeed, it's easy to see that the requirement that only nonnegative powers of the new coordinate a are present leads to the absence of all powers of z bigger than 4 in the second term and all powers of z bigger than 7 in the third term of RHS of (19), so that by Tate's algorithm[24], the second E_8 is unhiggsed. Also, if we count F-theory moduli, we'll find precisely the number equal to the number of instantons in the second E_8 , and they are obviously interpreted as corresponding to the position of point-like instantons.

The general picture which emerges is best illustrated by the following examples.

Consider a dual to the heterotic model with 12 instantons in the first E_8 and no instantons in the second E_8 . We know that the Calabi-Yau in this case has $(h_{11}, h_{21}) = (23, 143)$ and its dual polyhedron consists of:

- A set of points with $x_1 = 0$ which happens to be a dual polyhedron of the $K3$ exhibiting unbroken E_8 symmetry
- One point with $x_1 = -1$
- A set of points $(1, 0, 0, -1), (1, 1, 0, -1), \dots (1, 12, 0, -1)$

Now let's look at the dual to a heterotic model with 12 finite size instantons in the first E_8 and 12 small instantons in the second E_8 . The Calabi-Yau threefold turns out to have exactly the same Hodge numbers: $(h_{11}, h_{21}) = (23, 143)$ and its dual polyhedron consists of:

- Same set of points as before with $x_1 = 0$
- Same point with $x_1 = -1$
- Only one point with $x_1 = 1$: $(1, 0, 0, -1)$

This picture holds in general: take a $(d_1, 0)$ vacuum. It has an unbroken $G \times E_8$ gauge group where G corresponds to the maximally Higgsed case in the first E_8 . Compare it to any (d_1, d_2) vacuum where all d_2 instantons are point-like. The corresponding Calabi-Yau threefold is characterized by the same Hodge numbers, and the dual polyhedron differs by the absence of the points $(1, 24 - (d_1 + d_2) + 1, 0, -1)) \dots (1, 24 - d_1, 0, -1)$. However, the number of non-toric deformations is precisely equal to the number of missing points.

Thus, even though the polyhedra look different, the manifolds are actually the same¹. This is because we have ignored the non-toric deformations which are not visible as points in the polyhedra. In many cases, it is possible to ignore them because they are often zero. However, these non-toric deformations account for the observed discrepancy - the polyhedra obtained by unhiggsing E_8 have precisely as many non-toric deformations as missing tensor multiplets. Taking these into account, we are led to conclude that these polyhedra all describe the same manifold corresponding to a heterotic vacuum with 13 tensor multiplets and E_8 gauge symmetry.

In the same way we can construct duals to heterotic vacua with the first E_8 unhiggsed. This amounts to replacing u in (18) by v and writing (19) with appropriate power of the new coordinate.

We can also unhiggs both E_8 's which will result in the unbroken $E_8 \times E_8$ visible in the $K3$ subpolyhedron. The Hodge numbers of our Calabi-Yau are invariably $(43, 43)$ and there are 25 tensor multiplets, but not all of them are visible in the fan of the base, some being accounted for by the non-toric deformations.

4 Conclusion

In this paper we have considered F-theory duals of heterotic $E_8 \times E_8$ compactifications on $K3$ with instanton numbers (d_1, d_2) , and n_T tensor multiplets. The duals of these theories are conjectured to be F-theory compactifications on elliptic Calabi-Yau threefolds [26, 24]. This duality is related to the heterotic/type IIA dualities proposed in [1, 9]. It was observed in [18] that the sequences of reflexive polyhedra associated to these spaces are nested so

¹We are grateful to Paul Aspinwall for explaining this point to us.

as to reflect heterotic perturbative and non-perturbative processes. Note that the fact that these polyhedra are nested implies that the moduli spaces of the corresponding Calabi-Yau manifolds are connected [30]. The new contribution here is to show that the sequences of reflexive polyhedra in [18] can be obtained systematically using hypersurfaces in toric varieties, which are blow-ups of the \mathbb{F}_n models described in [24]. We also find that the blowups are encoded torically by extra points in the fan of the \mathbb{F}_n , which is visible in the dual polyhedra.

The models that we find would be the lowest members of chains of duals which can be obtained by unhiggsing in the spirit of refs. [13, 9, 18, 22].

We also show how to construct Calabi-Yau's corresponding to heterotic models by unhiggsing either, or both, E_8 's. We find that in such cases there are as many extra tensor multiplets as there were instantons in the unhiggsed E_8 , but not all of them appear as points corresponding to blowups of the fan of the \mathbb{F}_n in the dual polyhedra - some are encoded non-torically.

Acknowledgements

We would like to thank A. Font and J. Distler for useful discussions. We would also like to thank P. S. Aspinwall for pointing out an error in an earlier version. This work was supported in part by the Robert A. Welch Foundation and the NSF grant PHY-9511632.

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